II - Simplicial homology

PSL Week - Topological Data Analysis

Abstract

We introduce simplicial homology, the basic algebraic tool used in topological data analysis to detect connected components, loops, voids and higher dimensional holes.

We first define simplicial complexes, a combinatorial way to encode spaces using vertices, edges, triangles and their higher-dimensional analogues. We then construct homology groups over the field $\mathbb{Z}/2\mathbb{Z}$, which will be sufficient for most applications in this course. We briefly discuss coefficients in other fields and then generalize the construction to singular homology, which is defined for *any* topological space. Finally, we explain why continuous maps induce linear maps between homology groups and how this behaves with respect to composition.

Contents

1 Generalized triangulations

1.1 Simplicial complexes

Definition 1.1 (Geometric simplex)**.** Let $\mathcal{P} = \{p_0, \dots, p_k\} \subset \mathbb{R}^d$ be a finite set of points such that the vectors $p_1 - p_0, \dots, p_k - p_0$ are linearly independent (equivalently, the p_i do not lie in an affine subspace of dimension $< k$). The *k-dimensional simplex* (or *k-simplex*) spanned by $\mathcal{P}$ is

$$\sigma = [p_0, \dots, p_k] := \Big\{ \sum_{i=0}^{k} \lambda_i p_i : \lambda_i \geq 0, \ \sum_{i=0}^{k} \lambda_i = 1 \Big\}.$$

The points $p_0, \dots, p_k$ are the *vertices* of σ.

Remark 1.2. (a) σ is the convex hull of $\mathcal{P}$: the smallest convex subset of $\mathbb{R}^d$ containing the p_i.

(b) A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a filled triangle, a 3-simplex is a filled tetrahedron.

(c) The *faces* of $\sigma = [p_0, \dots, p_k]$ are the simplices spanned by non-empty subsets of $\{p_0, \dots, p_k\}$. For instance, the faces of a triangle are its three edges and its three vertices.

Definition 1.3 (Simplicial complex)**.** A (finite) *simplicial complex* $\mathcal{K}$ in $\mathbb{R}^d$ is a finite collection of simplices such that:

(i) if $\sigma \in \mathcal{K}$ and τ is a face of σ, then $\tau \in \mathcal{K}$;

(ii) the intersection of any two simplices of $\mathcal{K}$ is either empty or a common face of both.

Example 1.4. The left-hand side of Figure 1 shows a simplicial complex made of vertices, edges and triangles. The right-hand side shows a union of simplices that is *not* a simplicial complex because the intersection of two triangles is not a common face.

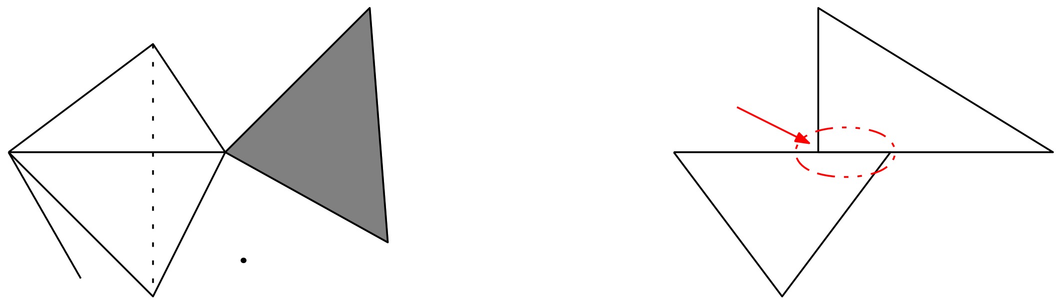

Figure 1: Left: a simplicial complex. Right: a union of simplices which is *not* a simplicial complex.

Definition 1.5 (Geometric realization)**.** Let $\mathcal{K}$ be a simplicial complex in $\mathbb{R}^d$. Its *geometric realization* is the subset

$$|\mathcal{K}| := \bigcup_{\sigma \in \mathcal{K}} \sigma \subset \mathbb{R}^d,$$

equipped with the subspace topology inherited from $\mathbb{R}^d$.

Remark 1.6. In practice we often do not distinguish between a simplicial complex and its geometric realization, and simply write $\mathcal{K}$ for both when there is no risk of confusion.

Definition 1.7 (Abstract simplicial complex)**.** Let $V = \{v_1, \dots, v_n\}$ be a finite set. An *abstract simplicial complex* $\tilde{\mathcal{K}}$ on V is a collection $\tilde{\mathcal{K}} \subset \mathcal{P}(V)$ of finite subsets of V such that:

(i) every singleton $\{v_i\}$ belongs to $\tilde{\mathcal{K}}$;

(ii) if $\sigma \in \tilde{\mathcal{K}}$ and $\tau \subset \sigma$, then $\tau \in \tilde{\mathcal{K}}$.

Elements of $\tilde{\mathcal{K}}$ play the role of simplices; they remember only the combinatorics, not the geometry.

One can show that any finite abstract simplicial complex admits a geometric realization in some $\mathbb{R}^D$, for D large enough. Furthermore, different realizations of the same abstract (combinatorial) simplicial complex have homeomorphic (geometric) realizations. For homology, we may freely switch between the combinatorial and geometric points of view.

1.2 Two important examples: Čech and Vietoris–Rips complexes

Simplicial complexes are used in TDA to build combinatorial models of a point cloud $\mathcal{P} \subset \mathbb{R}^d$. Two standard constructions are the Čech and Vietoris–Rips complexes.

Definition 1.8 (Čech complex)**.** Let $\mathcal{P} \subset \mathbb{R}^d$ be a finite set and $\alpha > 0$. The *Čech complex* $\mathrm{Cech}(\mathcal{P}, \alpha)$ has:

- vertex set $\mathcal{P}$;
- a simplex $[x_0, \ldots, x_k]$ whenever the intersection of the closed balls of radius α around the vertices is non-empty:

$$[x_0, \ldots, x_k] \in \mathrm{Cech}(\mathcal{P}, \alpha) \quad \Longleftrightarrow \quad \bigcap_{i=0}^{k} \overline{B}_\alpha(x_i) \neq \emptyset.$$

As α increases, we obtain a nested family of complexes $\mathrm{Cech}(\mathcal{P}, \alpha)$, called the *Čech filtration.*

Definition 1.9 (Vietoris–Rips complex)**.** Let $\mathcal{P} \subset \mathbb{R}^d$ be a finite set and $\alpha > 0$. The *Vietoris–Rips complex* $\mathrm{Rips}(\mathcal{P}, \alpha)$ has:

- vertex set $\mathcal{P}$;
- a simplex $[x_0, \ldots, x_k]$ whenever all pairwise distances are at most α:

$$[x_0, \ldots, x_k] \in \mathrm{Rips}(\mathcal{P}, \alpha) \quad \Longleftrightarrow \quad \|x_i - x_j\| \leq \alpha \ \text{ for all } i, j.$$

Again, as α increases we obtain the *Rips filtration* $\mathrm{Rips}(\mathcal{P}, \alpha)$.

Remark 1.10. The Čech complex at scale α is the nerve of the cover of $\bigcup_{x \in \mathcal{P}} \overline{B}_\alpha(x)$ by these balls. It is homotopy equivalent to that union of balls, which often approximates an underlying shape. The Rips complex is easier to compute (it uses only distances), and one has

$$\mathrm{Rips}(\mathcal{P}, \alpha) \subset \mathrm{Cech}(\mathcal{P}, \alpha) \subset \mathrm{Rips}(\mathcal{P}, 2\alpha),$$

so their homology is closely related.

2 Chains and simplicial homology

We next build homology groups for a finite simplicial complex $\mathcal{K}$. In this section, all coefficients are taken in the field $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, which avoids orientation issues.

2.1 Chains

Fix a finite simplicial complex $\mathcal{K}$. For each $k \geq 0$ let $\mathcal{K}_k$ denote the set of k-simplices of $\mathcal{K}$.

Definition 2.1 (Space of k-chains)**.** A *k-chain* on $\mathcal{K}$ (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is a formal linear combination

$$c = \sum_{i=1}^{p} \varepsilon_i \sigma_i,$$

where $\sigma_1, \ldots, \sigma_p$ are the k-simplices of $\mathcal{K}$ and $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. The set of all k-chains is denoted

$$C_k(\mathcal{K}) := \Big\{ \sum_{i=1}^{p} \varepsilon_i \sigma_i : \ \varepsilon_i \in \mathbb{Z}/2\mathbb{Z} \Big\}.$$

Addition and scalar multiplication are defined coefficientwise mod 2. Thus $C_k(\mathcal{K})$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$, with the k-simplices as a basis.

Geometrically, a k-chain is just the union of some k-simplices of $\mathcal{K}$: a coefficient 1 means that the simplex is present, and 0 means it is absent. Adding two chains corresponds to taking the symmetric difference of the corresponding families of simplices.

Example 2.2. Consider the complex $\mathcal{K}$ made of four vertices $1, 2, 3, 4$, five edges $[1, 2]$, $[1, 3]$, $[1, 4]$, $[2, 3]$, $[3, 4]$, and one triangle $[1, 3, 4]$, see Figure 2. The space $C_1(\mathcal{K})$ of 1-chains is the $\mathbb{Z}/2\mathbb{Z}$-vector space spanned by the four edges:

$$C_1(\mathcal{K}) = \operatorname{span}_{\mathbb{Z}/2\mathbb{Z}}\{[1, 2], [1, 3], [1, 4], [2, 3], [3, 4]\} \simeq (\mathbb{Z}/2\mathbb{Z})^5.$$

For instance, the chain $c = [a, b] + [b, c]$ corresponds to the union of the two edges ab and bc, and $c + [b, c] = [a, b]$ since $[b, c]$ cancels mod 2.

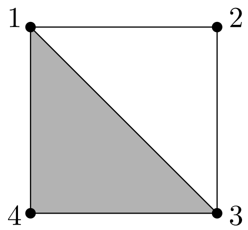

Figure 2: A 2-dimensional simplicial complex with 4 vertices, 5 edges, and 1 triangle.

2.2 Boundary operator

Definition 2.3 (Boundary of a simplex)**.** Let $\sigma = [v_0, \ldots, v_k]$ be a k-simplex of $\mathcal{K}$, $k \geq 1$. Its *boundary* is the $(k-1)$-chain

$$\partial\sigma := \sum_{i=0}^{k} [v_0, \ldots, \hat{v}_i, \ldots, v_k],$$

where the hat $\hat{v}_i$ means that v_i is omitted. Thus $\partial\sigma$ is the sum of all faces of σ of dimension $k - 1$.

Definition 2.4 (Boundary operator)**.** The *boundary operator* in degree k is the linear map

$$\partial_k : C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K})$$

defined by extending ∂ linearly from simplices to chains:

$$\partial_k\Big(\sum_i \varepsilon_i \sigma_i\Big) := \sum_i \varepsilon_i\, \partial\sigma_i.$$

We usually omit the index k and simply write ∂ when the degree is clear from the context.

Example 2.5 (Example of boundary operators)**.** Consider the simplicial complex $\mathcal{K}$ from Figure 2. In matrix form, its boundary operators are

$$\partial_2 = \begin{array}{c} \\ [12] \\ [13] \\ [14] \\ [23] \\ [34] \end{array} \begin{array}{c} [134] \\ \left(\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{array}\right) \end{array}, \qquad \partial_1 = \begin{array}{c} \\ [1] \\ [2] \\ [3] \\ [4] \end{array} \begin{array}{c} \begin{array}{ccccc} [12] & [13] & [14] & [23] & [34] \end{array} \\ \left(\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \end{array}.$$

Proposition 2.6 (Boundary of a boundary is zero)**.** *For all $k \geq 1$ one has*

$$\partial_{k-1} \circ \partial_k = 0.$$

Equivalently, the boundary of a boundary is always the zero chain.

Idea. It suffices to check the formula on a single k-simplex $\sigma = [v_0, \ldots, v_k]$ and extend linearly. Every $(k-2)$-face of σ appears exactly twice in $\partial(\partial\sigma)$ (once for each way of omitting two vertices), and since we work over $\mathbb{Z}/2\mathbb{Z}$ these contributions cancel mod 2. Thus $\partial(\partial\sigma) = 0$ for every simplex, hence $\partial \circ \partial = 0$ on all chains by linearity. □

2.3 Cycles, boundaries and homology groups

Definition 2.7 (Cycles and boundaries)**.** For $k \geq 0$:
the space of *k-cycles* is the kernel

$$Z_k(\mathcal{K}) := \ker\big(\partial_k : C_k(\mathcal{K}) \to C_{k-1}(\mathcal{K})\big),$$

i.e. the k-chains whose boundary is zero.
The space of *k-boundaries* is the image

$$B_k(\mathcal{K}) := \mathrm{Im}\big(\partial_{k+1} : C_{k+1}(\mathcal{K}) \to C_k(\mathcal{K})\big),$$

i.e. those k-chains that are the boundary of some $(k+1)$-chain.

The previous proposition implies that $B_k(\mathcal{K}) \subset Z_k(\mathcal{K})$: every boundary is a cycle. But not every cycle is a boundary; the homology groups measure the "difference".

Examples of chains, cycles and boundaries are given in Figure 3.

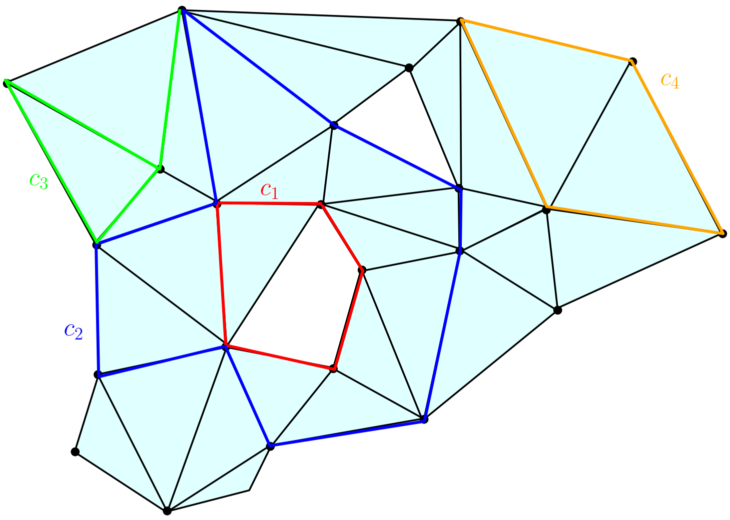

Figure 3: Some examples of chains, cycles and boundaries on a 2-dimensional complex $\mathcal{K}$: c_1, c_2 and c_4 are 1-cycles; c_3 is a 1-chain but not a 1-cycle; c_4 is a 1-boundary, namely the boundary of the 2-chain obtained as the sum of the two triangles surrounded by c_4; The cycles c_1 and c_2 span the same element in $H_1(\mathcal{K})$ as their difference is the 2-chain represented by the union of the triangles surrounded by the union of c_1 and c_2.

The linear spaces B_k and Z_k are subspaces of C_k, and according to Proposition 2.6, one has

$$B_k(\mathcal{K}) \subset Z_k(\mathcal{K}) \subset C_k(\mathcal{K}).$$

Definition 2.8 (Homology groups and Betti numbers)**.** The *kth homology group* of $\mathcal{K}$ (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is the quotient

$$H_k(\mathcal{K}) := Z_k(\mathcal{K})/B_k(\mathcal{K}).$$

Its elements are *homology classes*.
The *kth Betti number* of $\mathcal{K}$ is

$$\beta_k(\mathcal{K}) := \dim_{\mathbb{Z}/2\mathbb{Z}} H_k(\mathcal{K}).$$

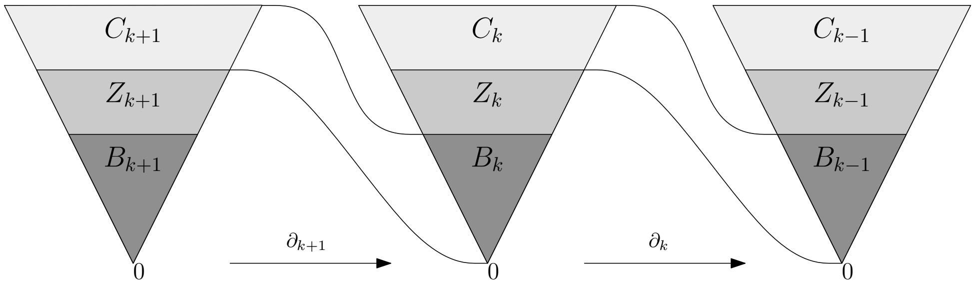

Figure 4: A chain complex, with the property $B_k(\mathcal{K}) \subset Z_k(\mathcal{K}) \subset C_k(\mathcal{K})$ dislayed. The composition of any two consecutive boundary maps is the zero map.

Example 2.9 (0-dimensional homology)**.** $C_0(\mathcal{K})$ is the vector space spanned by the vertices. The boundary map ∂_0 is zero, so $Z_0(\mathcal{K}) = C_0(\mathcal{K})$. A 0-boundary is the boundary of a 1-chain, i.e. a union of vertices that can be "paired up" by edges. One can show that

$$\beta_0(\mathcal{K}) = \text{number of connected components of } |\mathcal{K}|.$$

Thus H_0 encodes connectedness.

Example 2.10 (1-dimensional homology)**.** For a simplicial complex embedded in the plane, a 1-cycle is a union of edges with no boundary; geometrically, it is a union of closed polygonal curves. A 1-boundary is the boundary of a union of triangles. Thus $H_1(\mathcal{K})$ represents loops that do *not* bound any 2-dimensional region in the complex.

In the complex of Figure 2, the triangle $[1, 3, 4]$ fills in the cycle $[1, 3] + [1, 4] + [3, 4]$, so that cycle is a boundary and represents 0 in $H_1(\mathcal{K})$. On the other hand, if we add an extra square cycle with no triangles filling it, it would represent a non-trivial class in H_1.

3 A few words on natural generalizations

3.1 General coefficients

So far we have worked over the field $\mathbb{Z}/2\mathbb{Z}$. The construction generalizes to any field $\mathbb{F}$ (and more generally to abelian groups), at the cost of introducing orientations.

Remark 3.1 (Coefficients in a field)*.* Let $\mathbb{F}$ be a field (for instance $\mathbb{R}$ or $\mathbb{Z}/p\mathbb{Z}$ with prime $p \geq 2$). One can define chain groups $C_k(\mathcal{K}; \mathbb{F})$ as the $\mathbb{F}$-vector space spanned by the k-simplices, with *ordered* vertices to keep track of orientation. The boundary of a k-simplex is then an alternating sum of its oriented faces. One checks that $\partial^2 = 0$ still holds, and defines

$$H_k(\mathcal{K}; \mathbb{F}) := Z_k(\mathcal{K}; \mathbb{F})/B_k(\mathcal{K}; \mathbb{F}).$$

In this course we will mostly use $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ to avoid dealing with orientations.

3.2 Homology of continuous spaces

Simplicial homology is defined for spaces that come with a simplicial complex structure. To treat general metric spaces or topological spaces, we turn to *singular homology.*

Let Δ_k denote the standard k-simplex in $\mathbb{R}^{k+1}$, i.e. the convex hull of the standard basis vectors $e_0, \dots, e_k$.

Definition 3.2 (Singular simplex and singular chains)**.** Let X be a topological space.

- A *singular k-simplex* in X is a continuous map $\sigma : \Delta_k \to X$.

- A *singular k-chain* (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is a finite formal linear combination

$$c = \sum_{i=1}^{p} \varepsilon_i \sigma_i,$$

  where each σ_i is a singular k-simplex and $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z}$. The vector space of singular k-chains is denoted $C_k^{\text{sing}}(X)$.

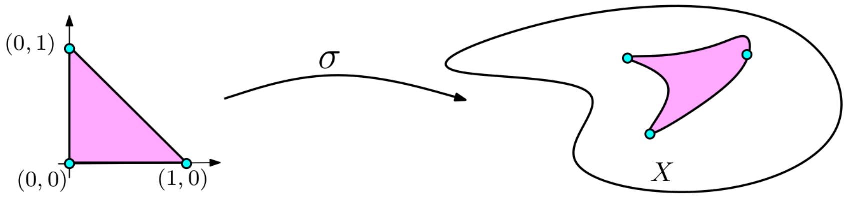

Figure 5: A singular k-simplex in a space X is a continuous map $\sigma : \Delta_k \to X$.

Definition 3.3 (Boundary of a singular simplex)**.** Let $\sigma : \Delta_k \to X$ be a singular k-simplex. Each $(k-1)$-face of Δ_k is itself a copy of Δ_{k-1}, obtained by setting one barycentric coordinate to zero. The *boundary* of σ is the $(k-1)$-chain

$$\partial\sigma := \sum_{i=0}^{k} \sigma|_{F_i},$$

where F_i is the ith face of Δ_k and $\sigma|_{F_i} : \Delta_{k-1} \to X$ is the restriction. Extending linearly defines boundary operators $\partial_k : C_k^{\text{sing}}(X) \to C_{k-1}^{\text{sing}}(X)$.

As in the simplicial case, one checks that $\partial_{k-1} \circ \partial_k = 0$ for all k, and defines cycles, boundaries and homology groups.

Definition 3.4 (Singular homology)**.** The *kth singular homology group* of X (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is

$$H_k^{\text{sing}}(X) := Z_k^{\text{sing}}(X)/B_k^{\text{sing}}(X),$$

where

$$Z_k^{\text{sing}}(X) := \ker \partial_k, \qquad B_k^{\text{sing}}(X) := \operatorname{Im} \partial_{k+1}.$$

Remark 3.5. If X is the geometric realization of a finite simplicial complex $\mathcal{K}$, there is a natural isomorphism

$$H_k(\mathcal{K}) \cong H_k^{\text{sing}}(|\mathcal{K}|).$$

We will not prove this here; roughly speaking, every continuous map from a simplex into $|\mathcal{K}|$ can be approximated by a simplicial map after subdividing the complex.

Example 3.6 (Some standard spaces)**.** For a field of coefficients, one can show that:

- $H_0^{\text{sing}}(X)$ has dimension equal to the number of path-connected components of X;
- $H_1^{\text{sing}}(S^1)$ has dimension 1 (one independent loop on the circle);
- $H_1^{\text{sing}}(B^2) = 0$ for the disk B^2 (all loops can be filled);
- $H_1^{\text{sing}}(T^2)$ for the torus has dimension 2 (two independent directions for loops).

These are the singular analogues of the simplicial examples from Chapter 1.

4 From continuous maps to linear morphisms

Homology is useful partly because it behaves well under continuous maps. Roughly speaking, a continuous map sends cycles to cycles and boundaries to boundaries, hence induces a linear map between homology groups. Moreover, composition of maps corresponds to composition of the induced maps.

We first describe this for simplicial maps, and then for general continuous maps using singular homology.

4.1 Simplicial maps

Definition 4.1 (Simplicial map)**.** Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes with vertex sets $V(\mathcal{K})$ and $V(\mathcal{L})$. A *simplicial map* $\varphi : \mathcal{K} \to \mathcal{L}$ is a map on vertices

$$\varphi : V(\mathcal{K}) \to V(\mathcal{L})$$

such that whenever $[v_0, \dots, v_k]$ is a simplex of $\mathcal{K}$, the set $\{\varphi(v_0), \dots, \varphi(v_k)\}$ spans a simplex of $\mathcal{L}$ (possibly of lower dimension).

A simplicial map sends simplices to simplices in a combinatorial way. It therefore induces linear maps

$$\varphi_{k\#} : C_k(\mathcal{K}) \to C_k(\mathcal{L})$$

by acting on each simplex, and these maps commute with the boundary operator:

$$\partial \circ \varphi_{k\#} = \varphi_{k-1\#} \circ \partial.$$

In other words, $\varphi_\#$ is a *chain map.*

Proposition 4.2 (Induced map in homology for simplicial maps)**.** *A simplicial map $\varphi : \mathcal{K} \to \mathcal{L}$ induces linear maps*

$$\varphi_* : H_k(\mathcal{K}) \to H_k(\mathcal{L})$$

for all $k \geq 0$, defined on homology classes by

$$\varphi_*([c]) := [\varphi_\#(c)].$$

Idea. Because $\varphi_\#$ commutes with ∂, it sends cycles to cycles and boundaries to boundaries. Therefore it induces a well-defined linear map on the quotient $Z_k/B_k = H_k$. □

Example 4.3 (Inclusion of a subcomplex)**.** If $\mathcal{A} \subset \mathcal{K}$ is a subcomplex, the inclusion $i : \mathcal{A} \hookrightarrow \mathcal{K}$ is a simplicial map. It induces maps

$$i_* : H_k(\mathcal{A}) \to H_k(\mathcal{K}).$$

For example, if $\mathcal{A}$ is a loop in $\mathcal{K}$ that bounds a 2-dimensional region in $\mathcal{K}$, then the corresponding class is non-trivial in $H_1(\mathcal{A})$ but maps to 0 in $H_1(\mathcal{K})$, because it becomes a boundary in $\mathcal{K}$.

4.2 Morphisms induced by continuous maps

For a general continuous map $f : X \to Y$ between topological spaces, we define the induced map using singular homology.

Definition 4.4 (Induced map on singular chains)**.** Let $f : X \to Y$ be continuous. For each singular simplex $\sigma : \Delta_k \to X$, define

$$f_\#(\sigma) := f \circ \sigma : \Delta_k \to Y.$$

Extend linearly to obtain a linear map

$$f_\# : C_k^{\text{sing}}(X) \to C_k^{\text{sing}}(Y).$$

Lemma 4.5. *For every k, the induced map $f_\#$ commutes with the boundary operators:*

$$\partial \circ f_\# = f_\# \circ \partial.$$

Proof. It suffices to check the equality on a singular simplex $\sigma : \Delta_k \to X$. On the one hand,

$$\partial(f_\# \sigma) = \partial(f \circ \sigma) = \sum_{i=0}^{k} (f \circ \sigma)|_{F_i}.$$

On the other hand,

$$f_\#(\partial\sigma) = f_\#\Big(\sum_{i=0}^{k} \sigma|_{F_i}\Big) = \sum_{i=0}^{k} f \circ (\sigma|_{F_i}) = \sum_{i=0}^{k} (f \circ \sigma)|_{F_i}.$$

The two expressions coincide, so $\partial \circ f_\# = f_\# \circ \partial$. □

Proposition 4.6 (Induced map in singular homology)**.** *Let $f : X \to Y$ be continuous. For each $k \geq 0$, the maps $f_\# : C_k^{\text{sing}}(X) \to C_k^{\text{sing}}(Y)$ induce well-defined linear maps on homology:*

$$f_* : H_k^{\text{sing}}(X) \to H_k^{\text{sing}}(Y),$$

defined by $f_*([c]) := [f_\#(c)]$.

Proof. Since $f_\#$ commutes with ∂, it sends cycles to cycles and boundaries to boundaries. Hence it induces a linear map on the quotient $Z_k^{\text{sing}}(X)/B_k^{\text{sing}}(X)$. □

Theorem 4.7 (Functoriality of homology)**.** *Let X, Y, Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then:*

(i) $(g \circ f)_* = g_* \circ f_*$ *as maps* $H_k^{\text{sing}}(X) \to H_k^{\text{sing}}(Z)$*;*

(ii) $(\text{id}_X)_* = \text{id}_{H_k^{\text{sing}}(X)}$.

Proof. On singular chains we have

$$(g \circ f)_\#(\sigma) = g \circ f \circ \sigma = g_\#(f_\#(\sigma)),$$

so $(g \circ f)_\# = g_\# \circ f_\#$ on $C_k^{\text{sing}}(X)$. Passing to homology classes gives

$$(g \circ f)_*([c]) = [(g \circ f)_\#(c)] = [g_\#(f_\#(c))] = g_*([f_\#(c)]) = g_*(f_*([c])).$$

The identity case is similar, with $\text{id}_\# = \text{id}$ on chains. □

Remark 4.8. Many references use the notation f_* for homology and f^* for cohomology. In these notes we follow this convention; the key point is that homology is a *functor* from topological spaces and continuous maps to vector spaces and linear maps.

Example 4.9 (Inclusion of a subspace)**.** Let $i : A \hookrightarrow X$ be the inclusion of a subspace. The induced map

$$i_* : H_k^{\text{sing}}(A) \to H_k^{\text{sing}}(X)$$

records how homology classes on A behave in X. For example:

- If $A = S^1$ is the boundary circle of a disk B^2, then the inclusion $i : S^1 \hookrightarrow B^2$ induces $i_* : H_1(S^1) \to H_1(B^2) = 0$. The fundamental loop on S^1 becomes null-homologous in B^2, because it bounds a 2-chain.

- If A is a loop embedded in a torus T^2 that goes around the hole, then the inclusion induces an injective map on H_1: that loop represents a non-trivial homology class in $H_1(T^2)$.

Theorem 4.10 (Homotopy and homology groups)**.** *If $f, g : X \to Y$ are homotopic maps (as defined in Chapter 1), then $f_* = g_*$ on homology.*

In particular, homotopy equivalent spaces have isomorphic homology groups:

$$X \simeq_h Y \quad \Longrightarrow \quad \beta_k(X) = \beta_k(Y) \quad \textit{for all } k \geq 0$$

The result above states that homology groups and Betti numbers are *homotopy invariants.* In general, beware with topological invariants: although they may allow you to check that two spaces are different, they can never tell you that two spaces have the same topology.

Remark 4.11 (Homology does not characterize homotopy)*.* The converse of the above result is *false.* That is, there exists spaces X and Y such that

- $\beta_k(X) = \beta_k(Y)$ for all $k \geq 0$;
- X and Y are *not* homotopy equivalent.

For instance, take the torus $X := S^1 \times S^2$ and the Poincaré sphere $Y := S^1 \wedge S^1 \wedge S^2$, obtained by gluing two circles S^1 to a sphere S^2 by one point. Then we have

$$\beta_0(X) = \beta_0(Y) = 1, \quad \beta_1(X) = \beta_1(Y) = 2, \quad \beta_2(X) = \beta_2(Y) = 1, \quad \beta_k(X) = \beta_k(Y) = 0, \quad k \geq 3$$

However, one can show that X and Y are not homotopy equivalent (which can be shown by constructing and computing other homotopy invariants called... The homotopy groups!).

5 Exercises

Basic exercises

Exercise 5.1 (Homology groups of some common spaces)**.** Propose triangulations of the following spaces, then compute their homology groups with coefficients in $\mathbb{Z}/2\mathbb{Z}$:

(a) the circle S^1,

(b) the disk B^2,

(c) the cylinder $S^1 \times [0,1]$,

(d) the 2-sphere S^2,

(e) the 3-ball B^3,

(f) the torus T^2, obtained from the unit square by identifying opposite edges.

Exercise 5.2 (Homology of the sphere S^d)**.** Let $d \geq 1$. Compute the homology groups $H_k(S^d; \mathbb{Z}/2\mathbb{Z})$ for all $k \geq 0$. (Hint: compare S^d with the boundary of a $(d+1)$-simplex.)

Advanced exercise

Exercise 5.3 (Structural properties of homology)**.** In this exercise we work with singular homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

(a) Let X and Y be topological spaces, and consider their disjoint union $X \sqcup Y$. Show that for every $k \geq 0$ there is a natural isomorphism

$$H_k(X \sqcup Y) \;\cong\; H_k(X) \oplus H_k(Y).$$

(Hint: use the fact that a singular simplex in $X \sqcup Y$ lands entirely in X or entirely in Y, so that $C_k^{\text{sing}}(X \sqcup Y) \cong C_k^{\text{sing}}(X) \oplus C_k^{\text{sing}}(Y)$.)

(b) Let X be a space with finitely many path-connected components $X_1, \ldots, X_r$. Use (a) and the identification $X \cong X_1 \sqcup \cdots \sqcup X_r$ to show that

$$H_0(X) \;\cong\; (\mathbb{Z}/2\mathbb{Z})^r.$$

In particular, $H_0(X)$ detects the number of path-connected components of X.

(c) Let X be any topological space and consider the projection

$$p : X \times [0,1] \longrightarrow X, \qquad p(x,t) = x,$$

and the inclusion

$$i : X \hookrightarrow X \times [0,1], \qquad i(x) = (x,0).$$

Show that $p \circ i = \mathrm{id}_X$ and that $i \circ p$ is homotopic to $\mathrm{id}_{X \times [0,1]}$ via a simple straight-line homotopy in the $[0,1]$-direction.

Using the fact that homotopic maps induce the same map in homology, deduce that for every $k \geq 0$,

$$H_k(X \times [0,1]) \;\cong\; H_k(X).$$

Solutions

Exercise 1 (Homology groups of some common spaces). We work over $\mathbb{Z}/2\mathbb{Z}$ throughout.

(a) *Circle* S^1. Triangulate S^1 by three vertices $\{1, 2, 3\}$ and three edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$. Then

$$C_0 \cong (\mathbb{Z}/2\mathbb{Z})^3, \qquad C_1 \cong (\mathbb{Z}/2\mathbb{Z})^3, \qquad C_k = 0 \text{ for } k \geq 2.$$

One checks that ∂_1 has rank 2 (e.g. by writing its 3×3 matrix), so

$$\dim H_0 = \dim \ker \partial_0 - \mathrm{rk}(\partial_1) = 3 - 2 = 1, \qquad \dim H_1 = \dim \ker \partial_1 - \mathrm{rk}(\partial_2) = 1 - 0 = 1.$$

Thus

$$H_0(S^1) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(S^1) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_k(S^1) = 0 \ (k \geq 2),$$

so $\beta_0 = 1$ (one component) and $\beta_1 = 1$ (one essential loop).

(b) *Disk* B^2. Triangulate B^2 by one triangle with vertices $\{1, 2, 3\}$. Then $C_0 \cong (\mathbb{Z}/2\mathbb{Z})^3$, $C_1 \cong (\mathbb{Z}/2\mathbb{Z})^3$, $C_2 \cong \mathbb{Z}/2\mathbb{Z}$.

The computation of H_0 is the same as for S^1: there is one connected component, hence

$$H_0(B^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

For H_1, there is one obvious 1-cycle (the boundary of the triangle), but it is also the boundary of the single 2-simplex, so $\mathrm{im}(\partial_2) \neq 0$ and in fact $H_1(B^2) = 0$. There are no 2-cycles (the unique 2-simplex has nonzero boundary), so $H_2(B^2) = 0$.

Thus B^2 has the same homology as a point: $\beta_0 = 1$ and $\beta_k = 0$ for $k \geq 1$.

(c) *Cylinder* $S^1 \times [0, 1]$. A direct (but heavy) way to answer this question is to triangulate the cylinder, for instance with 6 vertices and appropriate edges and triangles (see Figure 6), and then compute everything explicitly. Otherwise, one can use that $S^1 \times [0, 1]$ is homotopy equivalent to $S^1 \times \{0\}$, which gives

$$H_k(S^1 \times [0, 1]) \cong H_k(S^1).$$

Therefore

$$H_0 \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1 \cong \mathbb{Z}/2\mathbb{Z}, \quad H_k = 0 \ (k \geq 2).$$

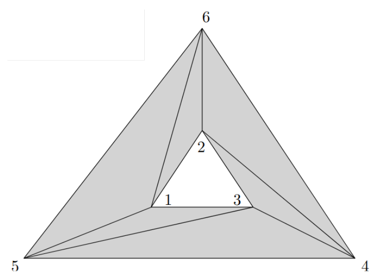

Figure 6: A triangulation of the cylinder, when seen as an annulus.

(d) *2-sphere* S^2. Triangulate S^2 as the boundary of a tetrahedron: 4 vertices, 6 edges and 4 triangular faces. One checks (by counting) that there is one connected component, no nontrivial 1-cycles (every loop bounds a union of triangles), and one independent 2-cycle (the whole surface).

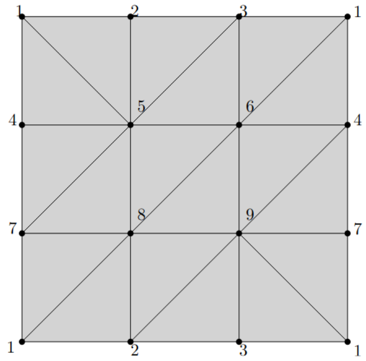

Figure 7: A triangulation of the 2-torus.

Formally:

$$H_0(S^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(S^2) = 0, \quad H_2(S^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

Thus $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = 1$.

(e) *3-ball B^3.* Triangulate the 3-ball by a tetrahedron (including its interior). This is convex, hence contractible. As with any other convex set, we have

$$H_0(B^3) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_k(B^3) = 0 \ (k \geq 1).$$

Again, B^3 has the same homology as a point.

(f) *Torus T^2.* The torus can be triangulated with a moderate number of vertices and triangles (see Figure 7). A convenient way to compute its homology is to think in terms of independent loops: the torus has one connected component, two independent 1-dimensional loops (around the "hole" and around the "tube"), and one 2-dimensional void enclosed by the surface.

The homology groups are

$$H_0(T^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(T^2) \cong (\mathbb{Z}/2\mathbb{Z})^2, \quad H_2(T^2) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_k(T^2) = 0 \ (k \geq 3).$$

Thus $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 1$.

Exercise 2 (Homology of the sphere S^d). We sketch an argument valid for all $d \geq 1$.

Consider a $(d+1)$-simplex Δ in $\mathbb{R}^{d+1}$ (the convex hull of $d+2$ affinely independent points). Its boundary $\partial\Delta$ is a union of d-simplices. One can check that $\partial\Delta$ is homeomorphic to S^d by radially projecting $\partial\Delta$ onto the sphere of radius 1 centered at a point inside Δ.

- The simplex Δ is convex, hence contractible: choosing a point p in its interior, the straight-line homotopy $H(x,t) = (1-t)x + tp$ deforms the identity to a constant map. Thus

$$H_0(\Delta) \cong \mathbb{Z}/2\mathbb{Z}, \qquad H_k(\Delta) = 0 \text{ for } k \geq 1.$$

- Apply the homology computation algorithm (or boundary-matrix reduction) to Δ and to $\partial\Delta$. The chain groups differ only in the top dimension: Δ has one extra $(d+1)$-simplex. Adding this last simplex kills exactly one d-cycle (its boundary), so the d-th Betti number of the boundary is one larger than that of Δ:

$$\beta_d(\partial\Delta) = \beta_d(\Delta) + 1 = 1.$$

In lower degrees $k < d$, the chain complexes of Δ and $\partial\Delta$ coincide in degrees $\leq k+1$, so the Betti numbers agree:

$$\beta_0(\partial\Delta) = \beta_0(\Delta) = 1, \quad \beta_k(\partial\Delta) = \beta_k(\Delta) = 0 \text{ for } 1 \leq k \leq d-1.$$

Since $\partial\Delta$ is homeomorphic to S^d, we conclude

$$H_0(S^d) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_d(S^d) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_k(S^d) = 0 \text{ for } 1 \le k \le d-1,\ k > d.$$

Exercise 3 (Structural properties of homology).

(a) A singular k-simplex in $X \sqcup Y$ is by definition a continuous map

$$\sigma : \Delta_k \longrightarrow X \sqcup Y.$$

Since Δ_k is connected and $X \sqcup Y$ is a disjoint union, the image of σ must lie entirely in X or entirely in Y. Thus any singular k-simplex in $X \sqcup Y$ can be viewed either as a simplex in X or as a simplex in Y, and never "mixes" the two.

Therefore every singular k-chain in $X \sqcup Y$ can be written uniquely as a sum of a chain supported in X and a chain supported in Y, and we obtain a natural vector space isomorphism

$$C_k^{\text{sing}}(X \sqcup Y) \;\cong\; C_k^{\text{sing}}(X) \oplus C_k^{\text{sing}}(Y).$$

Under this identification, the boundary operator on $C_k^{\text{sing}}(X \sqcup Y)$ is just the direct sum of the boundary operators on X and on Y. More precisely, if we identify a chain c on $X \sqcup Y$ with a pair (c_X, c_Y), then

$$\partial(c_X, c_Y) = (\partial c_X, \partial c_Y).$$

Thus the singular chain complex of $X \sqcup Y$ is the direct sum of the chain complexes of X and of Y:

$$C_*^{\text{sing}}(X \sqcup Y) \;\cong\; C_*^{\text{sing}}(X) \;\oplus\; C_*^{\text{sing}}(Y).$$

Homology commutes with taking direct sums of chain complexes: the cycles and boundaries split degreewise, and the quotient Z_k/B_k of a direct sum is the direct sum of the quotients. Concretely,

$$Z_k(X \sqcup Y) \cong Z_k(X) \oplus Z_k(Y), \qquad B_k(X \sqcup Y) \cong B_k(X) \oplus B_k(Y),$$

so

$$H_k(X \sqcup Y) = \frac{Z_k(X \sqcup Y)}{B_k(X \sqcup Y)} \;\cong\; \frac{Z_k(X) \oplus Z_k(Y)}{B_k(X) \oplus B_k(Y)} \;\cong\; \frac{Z_k(X)}{B_k(X)} \oplus \frac{Z_k(Y)}{B_k(Y)} = H_k(X) \oplus H_k(Y).$$

(b) If X has finitely many path-connected components $X_1, \dots, X_r$, then as a topological space X is homeomorphic to the disjoint union

$$X \;\cong\; X_1 \sqcup \cdots \sqcup X_r.$$

Homology is invariant under homeomorphism, so

$$H_0(X) \;\cong\; H_0(X_1 \sqcup \cdots \sqcup X_r).$$

Applying (a) repeatedly, we obtain

$$H_0(X_1 \sqcup \cdots \sqcup X_r) \;\cong\; H_0(X_1) \oplus \cdots \oplus H_0(X_r).$$

Each X_i is path-connected, so $H_0(X_i) \cong \mathbb{Z}/2\mathbb{Z}$ for every i. Therefore

$$H_0(X) \;\cong\; (\mathbb{Z}/2\mathbb{Z})^r.$$

In particular, the dimension of $H_0(X)$ over $\mathbb{Z}/2\mathbb{Z}$ is the number of path-connected components of X.

(c) The projection $p : X \times [0,1] \to X$, $p(x,t) = x$, and the inclusion $i : X \hookrightarrow X \times [0,1]$, $i(x) = (x,0)$, satisfy

$$p \circ i = \mathrm{id}_X.$$

On the other side, $i \circ p : X \times [0,1] \to X \times [0,1]$ is given by

$$(i \circ p)(x,t) = (x,0).$$

Define a homotopy

$$H : X \times [0,1] \times [0,1] \to X \times [0,1]$$

by

$$H\big((x,t),s\big) = (x,(1-s)t).$$

For $s = 0$,

$$H\big((x,t),0\big) = (x,t) = \mathrm{id}_{X\times[0,1]}(x,t),$$

and for $s = 1$,

$$H\big((x,t),1\big) = (x,0) = (i \circ p)(x,t).$$

Thus H is a homotopy between $\mathrm{id}_{X\times[0,1]}$ and $i \circ p$, so

$$\mathrm{id}_{X\times[0,1]} \simeq i \circ p.$$

We already have $p \circ i = \mathrm{id}_X$. Passing to homology and using that homotopic maps induce the same map in homology, we get

$$(\mathrm{id}_{X\times[0,1]})_* = (i \circ p)_* = i_* \circ p_*, \qquad (\mathrm{id}_X)_* = (p \circ i)_* = p_* \circ i_*.$$

Hence $p_* : H_k(X \times [0,1]) \to H_k(X)$ and $i_* : H_k(X) \to H_k(X \times [0,1])$ are inverse isomorphisms for every $k \geq 0$. In particular,

$$H_k(X \times [0,1]) \;\cong\; H_k(X)$$

for all k, as claimed.